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EQUILIBRIUM OF CAVITIES AND CRACKS-SLITS WITH OVERLAP AND OPENING DOMAINS IN AN ELASTIC MEDIUM*

R.V. GOL'DSHTEIN and YU.V. ZHITNIKOV

A class of three-dimensional problems in elasticity theory on the equilibrium of cracks-slits and cafvities (later also called cracks with an initial opening) is examined on the assumption that their surfaces can be superimposed on each other under the action of a system of volume forces. The boundaries of the overlap domains are not known in advance. The qualitative and extremal properties of the solutions that enable a correspondence to be established between solutions of problems for variations of the cavity domain and its initial opening (or the shape of the crack-slit) and, the external loads, in which enable the solutions of problems of the class under consideration and problems on the contact of two half-spaces to be compared are studied. A uniqueness theorem is formulated.

Proofs of all the assertions are obtained within the framework of a single approach by using asymptotic forms of the solutions near the boundaries of the overlap zones and congruence theorems /1/ (reflecting the property of positivity of the solutions of the problem under consideration). In this connection the asymptotic behaviour of the solution is first analysed in the neighbourhood of the overlap and opening zone boundaries of the cavity (crack-slit) surfaces. A class of functions that describes the initial cavity opening for which the solution near the boundaries of these domains is not singular is investigated. It is shown that the method of seeking the unkown overlap domain boundaries from the condition of no singularity while taking account of the proved assertions is equivalent to the method based on minimizing the elastic energy functional with constraints in the form of inequalities /2/.

The variational approach was applied to the investigation of spatial problems on cracks-slits with overlap domains /2/ and to the solution of problems of cavities /3/. The overlap domain boundaries as boundaries where the solution is not singular are constructed in the special case of the axisymmetric problem of a crack-slit in a layer /4/ and a cavity /5, 6/.

1. We consider the equilibrium of a linearly elastic isotropic space with a cavity whose section in the plane $x_3 = 0$ occupies a domain Ω. The distance between the cavity surfaces $u(x_1, x_2)$ is here a single-valued function of (x_1, x_2) and is small compared with the dimensions of Ω (a compressed cavity).

We will consider the conditions on the cavity surfaces in the $x_3 = 0$ plane. We will assume that the overlap domain is formed in the plane Ω under the effect of a system of volume forces that are symmetric about the plane $x_3 = 0$. The boundary conditions in the $x_3 = 0$ plane have the form

$$\sigma_{33}^{\pm} = 0\ (x_1, x_2) \in \Omega \setminus F;\ \sigma_{33}^{\pm} \leqslant 0,\ (x_1, x_2) \in F \tag{1.1}$$

$$\begin{gathered} u_3^+ - u_3^- = -u(x_1, x_2),\ (x_1, x_2) \in F \\ u_3^+ - u_3^- = 0,\ (x_1, x_2) \in R^2 \setminus \Omega,\ x_3 \to \pm 0 \\ u_3(x_1, x_2) \geqslant -u(x_1, x_2),\ (x_1, x_2) \in \Omega \end{gathered} \tag{1.2}$$

where F is the overlap domain of the cavity surfaces. Outside the domain Ω the external loads are given by a distribution of volume forces with density $\rho(x_1, x_2, x_3)$. If $u(x_1, x_2) \equiv 0$, then the cavity is converted into a crack-slit

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It is convenient to transfer from the system of external loads to the boundary conditions in the $x_3=0$ plane when solving the boundary value problem (1.1), (1.2). To do this, we find the stress distribution in a continuous body in the domain Ω: $-\sigma_{33}^\circ(x_1, x_2)$, by assuming that the corresponding elasticity theory problem for a body without a cavity is solved. Then, we append the stress $-\sigma_{33}^\circ(x_1, x_2)$ with opposite sign /2/ to the boundary conditions for the stresses in the domain Ω. After carrying out this procedure, the boundary conditions (1.1) take the form

$$\begin{aligned} &\sigma_{33}^\pm = \sigma_{33}^\circ(x_1, x_2), \ (x_1, x_2) \in \Omega \setminus F \\ &\sigma_{33}^\pm \leqslant \sigma_{33}^\circ(x_1, x_2), \ (x_1, x_2) \in F \end{aligned} \tag{1.3}$$

Later, by analogy with the boundary value problem (1.1), (1.2), we retain the designation "free surface" outside the domain $\Omega \setminus F$, and denote it by D.

2. We will examine the boundary value problem (1.2), (1.3). We set up the criterion governing the position of the overlap and opening domain boundaries. To do this, we will analyse the behaviour of the solution in the neighbourhood of the boundary as a function of the geometry of the initial opening $u(x_1, x_2)$.

We introduce a local XYZ coordinate system (the Z axis is directed along the tangent to the free surface boundary $\Omega \setminus F$, the Y axis is normal to the plane $x_3 = 0$; $x \leqslant 0$ corresponds to the domain $\Omega \setminus F$). Then by virtue of (1.2), (1.3), we have for $z = 0$

$$\begin{aligned} &u_y^+ - u_y^- = -u(x), \ x \geqslant 0; \ \sigma_{yy}^\pm(x, 0) = \sigma(x), \ x \leqslant 0, \\ &y \to \pm 0 \end{aligned} \tag{2.1}$$

where $\sigma(x)$ is determined by the external stress, and $u(x)$ is the given jump in the displacement. We assume that the functions $u(x), \sigma(x)$ satisfy the Hölder condition /7/, and are analytic in the neighbourhood of the point $x = 0$. By virtue of symmetry $\sigma_{zy}^\pm = \sigma_{zx}^\pm = 0$, $|x| < \infty$, $y = 0$.

The state of stress in the case of plane strain is described by the Kolosov-Muskhelishvili formulas /8, 9/ in the local coordinate system (μ is the shear modulus)

$$\begin{aligned} &\sigma_{xx} + \sigma_{yy} = 2[\Phi(\eta) + \Phi^*(\eta)], \ \eta = x + iy \\ &\sigma_{yy} - i\sigma_{xy} = \Phi(\eta) + \Omega(\eta^*) + (\eta - \eta^*)\Phi'^*(\eta) \\ &2\mu(u_x' + iu_y') = \varkappa\Phi(\eta) - \Omega(\eta^*) - (\eta - \eta^*)\Phi'^*(\eta) \\ &u_\alpha' = \partial u_\alpha/\partial x, \ \alpha = x, y; \ \mu = E/2(1+\nu), \ \varkappa = 3 - 4\nu \end{aligned} \tag{2.2}$$

Substituting the representations (2.2) into (2.1), we arrive at the following conjugate problem /9/:

$$\begin{aligned} &\Phi^+ - \Phi^- = ig_0'(x), \ x \geqslant 0; \ \Phi^+ + \Phi^- = \sigma(x), \ x \leqslant 0 \\ &g_0' = 2\mu(u_y^+ - u_y^-)/(\varkappa + 1) = -2\mu u(x)/(\varkappa + 1) = dg_0/dx \end{aligned} \tag{2.3}$$

Consider the canonical solution of problem (2.3): $x_0 = \eta^{1/2}$ for which $x_0^+ = x_0^-$, $x \geqslant 0$, $x_0^+ = -x_0^-$, $x \leqslant 0$, $y = 0$.

The solution of the boundary value problem (2.3) in the class of functions bounded at infinity and unbounded at the point $\eta = 0$ has the form /7/

$$\begin{aligned} &\Phi(\eta) = (M(\eta) + M_1(\eta) + B)/x_0(\eta) \\ &M(\eta) = \int_{-\infty}^0 \frac{\sigma(t)\, x_0(t)\, dt}{t - \eta}, \quad M_1(\eta) = \int_0^\infty \frac{ig_0'(t)\, x_0(t)\, dt}{t - \eta} \end{aligned}$$

(B is a constant determined from the conditions at infinity).

Consider the functions $w(\eta) = \sigma(\eta)x_0(\eta)$, $w_1(\eta) = ig_0'(\eta)x_1(\eta)$ where $x_1(\eta) = \eta^{1/2}$, $\sqrt{\eta} = \pm\sqrt{x}$, $x \geqslant 0$, $y \to \pm 0$ for the estimates $M(\eta)$, $M_1(\eta)$. A slit is drawn along the positive semi-axis in the XY plane. We obtain $(M - w)^+ = (M - w)^-$ and $(M_1 - w_1)^+ = (M_1 - w_1)^-$ as $y \to \pm 0$. Therefore, the functions $M(\eta) - w(\eta)$ and $M_1(\eta) - w_1(\eta)$ are analytic in the neighbourhood of the point $\eta = 0$; $M(\eta) - w(\eta) + M_1(\eta) - w_1(\eta)$ is an analytic function.

We therefore have near the point $\eta = 0$

$$\begin{aligned} &\Phi(\eta) = \sigma(\eta) + ig_0'(\eta)\,x_1(\eta)\,x_0^{-1}(\eta) + R_n(\eta)\,x_0^{-1}(\eta) \\ &R_n(\eta) = \sum_{k=1}^{n+1} A_k\eta^{k-1}\left(k - \frac{1}{2}\right) \end{aligned} \tag{2.4}$$

The stress and displacement distribution along the x axis in the neighbourhood of the point $x = 0$ takes the following form after substituting (2.4) into (2.2).

$$\begin{aligned}
&\sigma_{yy}^{\pm}(x, 0) = \sigma(x), \quad x \leqslant 0 \\
&\sigma_{yy}^{\pm}(x, 0) = \sigma(x) + {}^1/_2 x^{-1/2} A_1 + \ldots + (n - {}^1/_2) x^{n-3/2} A_n, \; x \geqslant 0 \\
&\sigma_{yy}(x, 0) \leqslant 0, \; x \geqslant 0; \; u_y(x, 0) = -u(x), \; x \geqslant 0 \\
&u_y(x, 0) = -u(x) + 2(1 - \nu)\mu^{-1}(r^{1/2} A_1 + \ldots \\
&\quad + (-1)^{n-1} r^{n-1/2} A_n), \quad x \leqslant 0, \quad r = |x|
\end{aligned} \tag{2.5}$$

It follows from (2.5) that the displacement and stress distributions along the x axis are related by means of the coefficients A_k.

We will analyse the stress and displacement distribution near the contact domain boundary. We will assume the equation of the cavity surface at $x \leqslant 0$ to have the form $u_0(x) = |x|^\alpha F_p(x) + B_1$, $F_p(x)$ is a polynomial of degree p, $0 < \alpha \leqslant 1$, i.e., a surface that is generally piecewise continuous at the point $x = 0$.

For $x \leqslant 0$ the cavity surfaces do not overlap, and consequently, the jump $u_y(x)$ in the displacement should be less in absolute value than the distance between them

$$|u_y(x, 0)| \leqslant u_0(x), \; x \leqslant 0 \tag{2.6}$$

Hence and from (2.5) it follows that if $p \geqslant 1$, then to satisfy (2.6) taking into account the fact that $\sigma_{yy} \leqslant 0$ in the contact domain $x \geqslant 0$, it is necessary that $A_1 = 0$ and $A_2 \leqslant 0$. If $p = 0$, then for ${}^1/_2 \leqslant \alpha \leqslant 1$ it is necessary that $A_1 = 0$ to satisfy (2.6) while $A_1 \neq 0$ and $A_1 < 0$ are necessary for $0 < \alpha < {}^1/_2$. In this case the stresses in the contact domain are compressive and singular.

As follows from the asymptotic form obtained, when $p \geqslant 1$ and $p = 0$, ${}^1/_2 \leqslant \alpha \leqslant 1$ the cavity surfaces interlock smoothly becuase of compression even when there are irregular points on an arbitary section of the surfaces. For $p = 0$, $0 < \alpha < {}^1/_2$ the compressive stresses near the breakpoint in the contact domain are singular, and complete closure of the surface is impossible for any finite A_1. Therefore, sections of the cavity surfaces can be indicated prior to the solution of the problem, which will not come into contact. If $u(x) \equiv 0$ for $x > 0$, then the asymptotic form constructed is the asymptotic form of the solution near an angle formed by surfaces with the equations $u(x) = |x|^\alpha F_p(x)$, $x \leqslant 0$ in a continuous material (the axis $x \geqslant 0$ is directed into the material).

The dependences on the parameters α, p obtained for the conditions of contact domain origination correspond to the results of numerical computations for cavities of elliptical planform /3/: for the initial opening of the cavity $(u(x_1, x_2) = b(1 - x_1^2/a^2 - x_2^2/b^2)^{\alpha/2})$ the contact domain abuts on the cavity boundary if $\alpha \geqslant 1$ and appears within the domain formed by a section through the cavity by the plane $x_3 = 0$ if $0 < \alpha < 1$.

Consider the case when the distance $u(x_1, x_2)$ between the cavity surfaces is a smooth function. Then to satisfy conditions (2.6) (there is an opening between the cavity surfaces) it is necessary that $A_1 > 0$ for $A_1 \neq 0$ and $A_2 \leqslant 0$ for $A_1 = 0$. For $A_1 \neq 0$, $\sigma_{yy}^{\pm}(x, 0) \geqslant 0$ according to (2.5), but this means that the cavity edges should be attracted into the continuity domain $x \geqslant 0$. If $x \geqslant 0$ is the contact domain, i.e., $\sigma_{yy}^{\pm}(x, 0) \leqslant 0$, then $A_1 \equiv 0$ is necessary and the stress distribution is not singular in the neighbourhood of the overlap domain boundary while the cavity surfaces interlock smoothly. A similar analysis is carried out for cracks-slits. In this case, it also follows from the conditions $u_y(x, 0) \geqslant 0$, $x \leqslant 0$ that $A_1 > 0$ for $A_1 \neq 0$ while we have $A_2 \leqslant 0$ for $A_1 = 0$. According to (2.5), the asymptotic form for $A_1 \neq 0$ corresponds to the condition $\sigma_{yy}(x, 0) > 0$, i.e., the crack edges should be attracted. Imposing the condition $\sigma_{yy}(x, 0) \leqslant 0$ in the contact domain $x \geqslant 0$, we have $A_2 \leqslant 0$ with the necessity that $A_1 = 0$.

For the domain Ω in the boundary value problem (1.1), (1.2) in the case when $u(x_1, x_2) = 0$, $(x_1, x_2) \in \Omega$, the boundary conditions in the free surface domain $\Omega \setminus F$ agree with the boundary conditions of the problem for a crack-slit occupying the domain $\Omega \setminus F$ in a continuous material if the stress intensity factor is zero on the boundary of the domain $\Omega \setminus F$ in the last problem. This enables the class of solutions of problems on the equilibrium of cracks-slits in a continuous material to be used to construct solutions of problems on the equilibrium of cracks-slits with overlap domains. The boundary Γ of the contact domains and the free surface in the problem for cracks-slits occupying the domain Ω is determined from the condition for seeking the contour of the cracks-slit of the domain $\Omega \setminus F$ on which $A_1(x_1^\circ, x_2^\circ) = 0$. Similarly for cavities the boundary Γ of the contact and free surfaces is determined by the condition of smooth interlocking of its surfaces

$$\lim \partial u_3(x_1, x_2)/\partial x = -\partial u(x_1^\circ, x_2^\circ)/\partial x, \quad (x_1, x_2) \to (x_1^\circ, x_2^\circ) \in \Gamma$$

The approach proposed for solving problems on cracks and cavities with overlap domains is analogous to the approach in /10, 11/ for the consideration of equilibrium cracks. The role of the adhesive force is played here by the compressive forces in the contact domain /11, 12/.

3. We will examine the question of the uniqueness of the solution of boundary value

problem (1.1), (1.2). We assume that the solution of the mixed problem of elasticity theory is unique for a half-space with a line of separation of the boundary conditions Γ /13/. In this case the solution of the problem about a cavity with overlap and opening domains subjected to a system of volume forces is unique.

Proof. We will assume that two opening domains D_1 and D_2 exist with displacement jumps $u_3^{(1)}, u_3^{(2)}$ and state of stress $\sigma_{ik}^{(1)}, \sigma_{ik}^{(2)}$ subjected to a given system of volume forces. The domains D_1, D_2 cannot coincide because of the assumed uniqueness of the mixed problem of elasticity theory with the separation boundary Γ. Consequently, their intersection $D_0: D_1 \cap D_2$ is non-empty. By virtue of the conditions in the overlap domain (1.1) we have

$$\sigma_{33}^{(1)} \leqslant 0, \ (x_1, x_2) \in D_2 \setminus D_0; \ \sigma_{33}^{(2)} \leqslant 0, \ (x_1, x_2) \in D_1 \setminus D_0 \tag{3.1}$$

We subtract the quantities corresponding to the second and first states and consider the upper half-space. Taking account of (1.2), (1.3), (3.1) in the plane $x_3 = 0$, we obtain

$$\begin{aligned} &v_3^+ = u_3^{(1)} - u(x_1, x_2) \geqslant 0, \quad \Sigma_{33} = -\sigma_{33}^{(2)} \geqslant 0, \quad (x_1, x_2) \in D_1 \\ &v_3^+ = u_3^{(1)} - u_3^{(2)}, \quad \Sigma_{33} = 0, \quad (x_1, x_2) \in D_0 \\ &v_3^+ = -u_3^{(2)} - u(x_1, x_2) \leqslant 0, \quad \Sigma_{33} = \sigma_{33}^{(1)} \leqslant 0, \quad (x_1, x_2) \in D_2 \\ &v_3 = 0, \quad (x_1, x_2) \in R^2 \setminus D_1 \cup D_2 \\ &\Sigma_{ik} = \sigma_{ik}^{(1)} - \sigma_{ik}^{(2)}, \quad v_i = u_i^{(1)} - u_i^{(2)} \end{aligned} \tag{3.2}$$

The elastic energy for the state of stress Σ_{ik} for given stresses and displacements on the boundary is

$$W = -\frac{1}{2} \int_{R^2} \Sigma_{33} v_3 n_3 \, dx_1 \, dx_2 \leqslant 0$$

which is impossible /13/. Therefore, $u_3^{(1)} = u_3^{(2)}$ and the uniqueness of the solution is proved.

An analogous assertion holds for a cracks-slit $(u(x_1, x_2) = 0, (x_1, x_2) \in \Omega)$ with overlap and adhesion domains.

Corollary 1. We consider a cavity (crack-slit) whose section by the plane $x_3 = 0$ occupies the domain Ω along which an overlap domain F is formed under the action of a system of volume forces. We now consider the very small cavity (crack-slit) subjected to the very same system of volume forces but with the overlap domain $F_1 \subset F$ obtained because of the application of additional forces to the cavity (crack-slit) surfaces in F_1. In this case there should be the subdomains $F_1' \subset F_1$, where $\sigma_{33}(x_1, x_2) \geqslant 0, (x_1, x_2) \in F_1'$ in the domain F_1. If the domains F_1' and D_1 have common boundary sections, then according to the asymptotic form (2.5) the solution is singular along these sections.

4. Using the asymptotic form of the solution near the free surface, Corollary 1 and the property of positivity of the solution /1/, an assertion can be proved that enables estimates to be made of solutions of problems for cracks-slits and cavities with overlap domains.

Assertion 1. We assume that cavity (cracks-slit) surfaces subjected to compressive loads overlap along the part $F_1 (\partial F_1 = \Gamma_1)$ of the domain Ω which is a section through the cavity by the plane $x_3 = 0$. We consider the very same cavity (cracks-slit) with the overlap domain $F_2, F_2 \supset F_1$ obtained because of the application of additional forces in F_2. If the boundary Γ_2 has a point of tangency M with Γ_1, then the distribution of normal displacement jumps of the cavity surfaces in the second case has the form $w_y(x) = -(u(x) - B_2 |x|^{3/2}), x \leqslant 0, B_2 \geqslant 0$ in the domain $\Omega \setminus F_2$ near the point of tangency M in the local XYZ coordinate system and the stress distribution $\sigma_{yy}(x) = -B_2 x^{1/2}, x \geqslant 0$ is not singular ($u(x)$ is the initial distance between the cavity surfaces, $u_y(x), w_y(x)$ are displacement jump components of the cavity surfaces, respectively, for the overlap domains F_1 and F_2, $u(x_1, x_2) \equiv 0, (x_1, x_2) \in \Omega$ is a cracks-slit).

Indeed, according to (2.5), under the assumption that $B_1 \neq 0$ when additional loads are applied to the cavity surfaces the displacement jump distribution $w_y(x)$ has the form $w_y(x) = -(u(x) - B_1 |x|^{1/2}), x \leqslant 0, B_1 \geqslant 0$ in the local coordinate system. Since the displacement jump distribution $u_y(x)$ in the first case has the form $u_y(x) = -(u(x) + A_2 |x|^{3/2}), x \leqslant 0, A_2 \leqslant 0$, then

$$|u_y(x)| > |w_y(x)| \tag{4.1}$$

We will show that under the assumptions made we can arrive at the opposite inequality. According to Corollary 1, subdomains $F'_2 \subset F_2$ on which $\sigma_{yy}(x, z) \geqslant 0$ should exist in the overlap domain F_2. We consider a displacement increment when the stresses in these subdomains F'_2 are removed. To do this, we will represent the problem in the form of a superposition of the original problem for the domain $\Omega \setminus F_2$ and for the domain $D_2 \cup F'_2$. There are no stresses in the domain D_2 while stresses with oppositve sign (tensile) are added in F'_2. According to the positivity property of the displacement jump /1/, we obtain that $\delta u_3 > 0$, u_3 grows, and $(x_1, x_2) \in D_2 \cup F'_2$. Let $F_2 \to F_1, \Gamma_2 \to \Gamma_1$ to increase D_2 because of the opening of the domain F'_2. By the uniqueness theorem $F_2 \to F_1$, $D_2 \to D_1$. Since the displacement jump grows /1/ because of the application of tensile stresses, we obtain $|u_y(x)| > |w_y(x)|$ in the neighbourhood of the point of tangency M, which contradicts the original assumption (4.1).

Assertion 1 permits the construction of estimates of solutions of problems on the equilibrium of cavities and cracks-slits with overlap domains and unknown boundary locations for these domains by using solutions of the corresponding problems for simpler geometric cavities (slits).

We consider the equilibrium of a cracks-slit occupying the domain Ω of the plane $x_3 = 0$ in an infinite medium under the action of volume forces. The slit surfaces can overlap in a certain domain $F \subset \Omega$ not known in advance and can open in a domain $D \subset \Omega$.

Assertion 2. If an opening domain D_1 is formed on a cracks-slit Ω_1 lying in Ω, $\Omega_1 \subset \Omega$ under the action of the same system of volume forces, then it lies within the domain D, $D_1 \subset D$.

We assume that the domain D_1 is not contained in D. According to Corollary 1, a part F'_1 on which $\sigma_{33}(x_1, x_2) \geqslant 0, (x_1, x_2) \in F'_1$ should be in the domain F_1. We expand the domain D_1 by adding the domain F'_1 by applying tensile stresses in F_1'. According to the congruence principle /1/, the displacement jump u_3 in the domain D_1 grows. Therefore, by expanding the opening domain D_1 different from D, we obtain a new opening domain which contradicts the uniqueness theorem. Therefore, it is necessary that $D_1 \subset D$. The assertion is proved.

Assertion 2 formulated for the overlap domains is proved in /2/ on the basis of variational inequalities.

We now prove two variational assertions for a crack-slit with an overlap domain, which are also a result of the asymptotic form obtained and the congruence theorem /1/.

Assertion 3. The displacement jump $u_3 = u_3^+ - u_3^-$ obtained for the cracks-slits $\Omega_1 \subset \Omega$ with opening domains $D_1 (u_3(x_1, x_2) \geqslant 0, (x_1, x_2) \in D_1)$ reaches a maximum at each point of the opening domain $D \subset \Omega$ in this field of external loads, outside of which the inequality $\sigma_{33}(x_1, x_2) \leqslant 0$, $(x_1, x_2) \in \Omega \setminus D$, is satisfied, i.e., in the solution of the boundary value problem (1.1), (1.2) under the conditions $u(x_1, x_2) \equiv 0$, $u_3(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \Omega$.

We will first show that the displacement jump reaches an extremum in the domain D under the condition $u_3(x_1, x_2) \geqslant 0, (x_1, x_2) \in \Omega$. We vary the free surface domain D near the boundary of D.

We consider the case when the domain D is expanded $\delta D \subset D$. To determine the increment of the displacement jump $u_3(x_1, x_2)$ during expansion of the domain D, we represent the boundary value problem (1.1), (1.2) as the superposition of two problems: the original for the domain D with a given system of volume forces and for the domain $D \cup \delta D$ in which there is load on D, and the stresses $\delta\sigma_{33}^{\pm}$ which were on the section δD in the continuous material but with opposite sign (superposition of the solutions), are appended on the section δD.

The asymptotic behaviour of the stresses near the boundary of the domain D outside of which $\sigma_{33}^{\pm}(x_1, x_2) \leqslant 0$, $(x_1, x_2) \in \Omega \setminus D$ is determined by expression (2.5), for $A_1 = 0$, $\sigma(x) = 0$, $x \leqslant 0$, $u(x) = 0$, $x \geqslant 0$, and $A_2 \leqslant 0$. Therefore, upon expansion of the domain D the stresses are $\delta\sigma_{33}^{\pm}(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \delta D$, and according to /1/ the displacement jump is $\delta u_3 \leqslant 0$ for any point $(x_1, x_2) \in D \cup \delta D$.

The inequality obtained $u_3 = \delta u_3 \leqslant 0, (x_1, x_2) \in \delta D$ corresponds to penetration of one crack-slit surface into another in the neighbourhood of the boundary of the domain D, which is impossible. Therefore, expansion of the domain D in the overlap domain is not allowable because of the constraint $u_3(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \Omega$.

We consider the case of shrinkage of the domain D, $\delta D \subset D$. It is here necessary to apply forces attracting one crack surface to the other, i.e., $\delta\sigma_{33}^{\pm} \geqslant 0$, in the section δD, and according to /1/, $\delta u_3 \leqslant 0$ at any point of the domain D. Therefore, the displacement jump does not grow when the free boundary varies. We will show that it reaches the maximum value at each point of the domain D outside of which $\sigma_{33}^{\pm}(x_1, x_2) \leqslant 0, (x_1, x_2) \in \Omega \setminus D$.

We consider different opening domains $D_1 \subset \Omega_1$ being formed in the cracks-slits $\Omega_1 \subset \Omega$. According to Assertion 2, all these domains are contained in D. According to Corollary 1, along the boundaries of these domains stress singularity sections exist that differ from the boundary of the domain Ω. Expanding the domain D_1 to the domain D along these sections according to the congruence theorem /1/ we find that the displacement jump u_3 grows at each point of the domain D_1, and, therefore, reaches a maximum upon coincidence with D.

Therefore, the displacement jump $u_3(x_1, x_2)$ determined in a set of opening domains $D_1(u_3(x_1, x_2) \geqslant 0, (x_1, x_2) \in D_1)$ reaches a maximum in this load field in the domain $D \subset \Omega$ outside of which $\sigma_{33}(x_1, x_2) \leqslant 0, (x_1, x_2) \in D$. Assertion 3 is proved.

A direct corollary of Assertion 3 is the attainment of a maximum of the volume of the crack-slit opening for the domain $D \subset \Omega$ outside which the inequality $\sigma_{33}^{\pm}(x_1, x_2) \leqslant 0$, $(x_1, x_2) \in \Omega \setminus D$ is satisfied under the condition $u_3(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \Omega$, i.e., in solutions of the boundary value problem (1.1) /14/.

We consider the change in elastic energy during variation of the loads and the free surface domain D in a crack-slit Ω. We perform the analysis for a body of finite volume V° containing the crack Ω to whose surface Σ the given loads are applied.

Let the stress and strain in state one be $\sigma_{ik}^{(1)}$ and $\varepsilon_{ik}^{(1)}$, and in state two $\sigma_{ik}^{(2)}$ and $\varepsilon_{ik}^{(2)}$. The Change in the elastic has the form /13/

$$\Delta W = \frac{1}{2}\int_V (\sigma_{ik}^{(2)}\varepsilon_{ik}^{(2)} - \sigma_{ik}^{(1)}\varepsilon_{ik}^{(1)})\, dV^\circ = W_2 - W_1 \tag{4.2}$$

where W_i is the energy in the i-th state, $i = 1, 2$. We use the Betti theorem /13/

$$\int_V (\sigma_{ik}^{(2)}\varepsilon_{ik}^{(1)} - \sigma_{ik}^{(1)}\varepsilon_{ik}^{(2)})\, dV^\circ = 0$$

Appending this relationship to the right side of (4.2), we obtain after changing to a surface integral

$$\Delta W = I_\Sigma [T_i (u_i^{(2)} - u_i^{(1)})] + I_{D\cup\delta D} [(\sigma_i^{(2)} + \sigma_i^{(1)}) (u_i^{(2)} - u_i^{(1)})]$$
$$I_s[fg] = \frac{1}{2}\int_s fg\, ds$$

where Σ is the surface enclosing the volume V°, and T_i is the vector of the forces applied to the surface Σ. On the other hand, according to the Clapeyron theorem

$$\Delta W = I_\Sigma [T_i (u_i^{(2)} - u_i^{(1)})] - I_{D\cup\delta D} [u_i^{(2)}\sigma_{i3}^{(2)} n_3] + I_D [u_i^{(1)}\sigma_{i3}^{(1)} n_3]$$

Eliminating the integral over Σ from the last two relationships, we obtain

$$\Delta W = - I_D [(u_i^{(2)} + u_i^{(1)}) (\sigma_{3i}^{(2)} - \sigma_{3i}^{(1)}) n_3] - \\ I_{\delta D} [u_i^{(2)} (\sigma_{3i}^{(2)} - \sigma_{3i}^{(1)}) n_3] - I_{\delta D} [u_i^{(1)} (\sigma_{3i}^{(2)} + \sigma_{3i}^{(1)}) n_3] \tag{4.3}$$

The volume V° and the surface Σ are not in the final result (4.3), which enables it to be used even to compute the energy changes of an infinite space. The relationship (4.3) enables the change in elastic energy to be analysed as a function of the change in stress and dimensions of the plane domain D.

We apply the result obtained to the problem of the equilibrium of a crack-slit Ω with an overlap domain F and a free surface D. We will examine the change in elastic energy during variation of the overlap zone boundaries in the boundary value problem (1.2), (1.3). In this case, according to (3.3) we obtain

$$\Delta W = - I_{\delta D} [u_i^{(2)} (\sigma_{3i}^{(2)} - \sigma_{3i}^{(1)}) n_3] - I_{\delta D} [u_i^{(1)} (\sigma_{3i}^{(2)} + \sigma_{3i}^{(1)}) n_3] \tag{4.4}$$

The two components in the expression for the elastic energy increment (4.4) correspond to expansion and shrinkage of the domain D. Indeed, if the domain D is expanded, then $u_i^{(1)}(x_1, x_2) = 0$, $(x_1, x_2) \in \delta D$ and the second component in (4.4) must be excluded. If the domain D is shrunk, then $u_i^{(2)}(x_1, x_2) = 0$, $(x_1, x_2) \in \delta D$ and the first component is eliminated. Denoting the displacement of the contour of the domain D in the normal direction by $\delta L(x_1, x_2)$ and assuming that the contour is smooth, we write the relationship (3.4) in the local XYZ coordinate system (the x axis is in the domain D and is directed along the normal to the contour, $x \leqslant 0$ corresponds to the domain D, and dl is an element of the arc of the contour)

$$\Delta W = -\frac{1}{2}\int_\Gamma \Big(\int_0^{\delta L} u_i^{(2)} (\sigma_{3i}^{(2)} - \sigma_{3i}^{(1)}) n_3\, dx\Big) dl, \quad \delta L > 0 \tag{4.5}$$

$$\Delta W = -\frac{1}{2}\int_\Gamma \Big(\int_{\delta L}^0 u_i^{(1)} (\sigma_{3i}^{(2)} + \sigma_{3i}^{(1)}) n_3\, dx\Big) dl, \quad \delta L < 0$$

We will calculate the energy change during variation of the domain D. To do this we substitute the asymptotic representation of the stresses and the displacement jump (2.5) in the local coordinate system in the neighbourhood of the boundary of the domain D

$$\Delta W = \frac{(1-\nu)\pi}{\mu}\int_\Gamma \Big[A_1^{(2)}A_1^{(1)}\delta L + (A_2^{(1)}A_1^{(2)} - A_1^{(1)}A_2^{(2)})\frac{3}{16}\delta L^2 + \ldots\Big]\, dl \tag{4.6}$$

We investigate (4.6) during the passage from different states. If $A_1^{(1)} \neq 0$, $A_1^{(2)} \neq 0$, and therefore, $A_1^{(1)} > 0$, $A_1^{(2)} > 0$ for the cracks-slit, then the change in elastic energy is determined by the first approximation in δL

$$\Delta W = \frac{(1-\nu)\pi}{\mu}\int_\Gamma A_1^{(2)}A_1^{(1)}\delta L\, dl$$

i.e., the elastic energy grows as the crack expands: $\delta W > 0$. In the limit $A_1^{(2)} \to A_1^{(1)} = A_1$ as $\delta L \to 0$, the expression for the change in elastic energy agrees with the Irwin formula if we set $A_1 = K_1/\sqrt{2\pi}$ where K_1 is the stress intensity factor /8, 9/.

Therefore, the elastic energy grows monotonically as the domain outside of which the constraint $\sigma_{33}^{\pm}(x_1, x_2) \leqslant 0$ is not satisfied is varied.

According to the assertion proved, the domains D_1 for which $u_3 \geqslant 0$, $(x_1, x_2) \in D_1$ lie in D, $D_1 \subset D$ and have sections of the boundary along which the stresses are singular, and do not agree with the boundaries of Ω. By expanding the domain D along these boundaries, we obtain that the elastic energy grows according to the preceding, and reaches a maximum on the boundary of the domain D on which $A_1 = 0$. According to the asymptotic form (2.5) and Assertion 3, for $\delta L < 0$, $A_1^{(2)} < 0$, $A_2^{(1)} < 0$, $A_1^{(1)} = 0$. Substituting the relationship (2.5) for the coefficients of the expansion into (4.6), we obtain that $\delta W < 0$ for $\delta L < 0$, i.e., W decreases. The variation of $\delta L > 0$ for the domain D is not allowable because of the constraint $u_3(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \Omega$.

The following can therefore be formulated

Assertion 4. The elastic energy functional

$$W = -\frac{1}{2}\int_{D_1} u_3\sigma_{33}n_3\,ds$$

calculated in the set of domains $D_1 \subset \Omega$ on which $u_3 \geqslant 0$, $(x_1, x_2) \in D_1$ in a given external load field reaches its maximum in the domain $D \subset \Omega$ outside which the inequality $\sigma_{33}^{\pm}(x_1, x_2) \leqslant 0$, $(x_1, x_2) \in \Omega \setminus D$ is satisfied under the condition $u_3(x_1, x_2) \geqslant 0$, $(x_1, x_2) \in \Omega$.

Assertion 4 has been formulated /2/ and proved by other means. The proof of Assertion 4 on the basis of the congruence theorem and the kind of asymptotic form set up above for solving the problem on the equilibrium of cracks near the boundary of the overlap zone of its surfaces simultaneously sets up the equivalence of the approaches to the solution of problems of a crack-slit with an overlap domain by a variational method and by constructing a non-singular solution. The non-singularity of the solution on the boundary of the free surface and contact domains follows from the requirement for the elastic energy functional to reach a maximum, and the maximum of the elastic energy follows from the requirement of non-singularity of the solution.

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